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Symmetry properties of s-classified SU(3) $3j$-, $6j$- and $9j$-symbols

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Abstract. Starting from the s-classified SU(3) Clebsch-Gordan coefficients introduced previously, the s-classified SU(3) $3j$-, $6j$- and $9j$-symbols are constructed and examined. They satisfy simple symmetry relations similar to those valid for the SU(2) case.

1. Introduction

In a previous paper the s-classified SU(3) Clebsch-Gordan coefficients have been introduced (Pluhař *et al* 1986, hereafter referred to as I). The reduced states obtained by their use are classified by eigenvalues s of an appropriate labelling operator constructed from representation generators. The coefficients satisfy simple symmetry relations analogous to those of their SU(2) counterparts.

In this paper the corresponding s-classified SU(3) $3j$-, $6j$- and $9j$-symbols are briefly examined. Their properties follow the general pattern established for an arbitrary group by Derome, Sharp and Butler (Derome and Sharp 1965, Derome 1966, Butler 1975). However, the symmetries of the s-classified Clebsch-Gordan coefficients allow important simplifications of the general formulae for the present case. This applies in particular to the symmetry relations between the s-classified $3nj$-symbols examined, which turn out to be simpler than those of the $3nj$-symbols in current use (cf Draayer and Akiyama 1973, Millener 1978).

2. Conventions and notation

The conventions introduced in I will be followed. The irreducible representations (IRs) of SU(3) will be specified by their highest weights (ab); the canonical basis states of the IRs will be specified by their hypercharge y, isospin i and isospin projection i_z. The notation will be simplified by employing the abbreviations

$$\begin{aligned} &j=(ab) \qquad m=(yii_z) \\ &\bar{j}=(ba) \qquad \bar{m}=(-yi-i_z) \qquad \bar{s}=-s \\ &(-1)^j=(-1)^{a+b} \qquad (-1)^m=(-1)^{\frac{3}{2}y+i_z}. \end{aligned} \tag{2.1}$$

The quantum numbers $\bar{j}$, $\bar{m}$ and $\bar{s}$ will be referred to as conjugates of j, m and s, respectively. The particular j and m numbers referring to the one-dimensional IR will be designated by the symbol 0. The dimension of the IR of highest weight (HW) j will be denoted by $|j|$. The eigenvalues of the Casimir operators of order p for the IR of HW j will be denoted by $f_p(j)$, $p=2,3$ (see I (2.3) and (2.4)).

In the new notation, the basic symmetry relations of the s-classified SU(3) Clebsch-Gordan coefficients (CGCs) take the form (see I (5.3)):

$$\begin{aligned}(j_1m_1j_2m_2|\bar{j}_3\bar{m}_3s) &= (j_2m_2j_1m_1|\bar{j}_3\bar{m}_3\bar{s})(-1)^{j_1+j_2+j_3}\\ &= (j_1m_1j_3m_3|\bar{j}_2\bar{m}_2\bar{s})(-1)^{j_1+m_1}(|j_2|/|j_3|)^{1/2}\\ &= (\bar{j}_1\bar{m}_1\bar{j}_2\bar{m}_2|j_3m_3\bar{s})(-1)^{j_1+j_2+j_3} \qquad (2.2)\end{aligned}$$

and the expression for the CGCs with $j_3=0$ (and, necessarily, $j_2=\bar{j}_1$, $m_3=0$ and $s=2f_3(j_1)$) becomes (see I (4.8))

$$(j_1m_1j_2m_2|00s)=\delta_{m_1\bar{m}_2}\frac{(-1)^{j_1+m_1}}{\sqrt{|j_1|}}. \qquad (2.3)$$

All s-classified SU(3) CGCs and $3nj$-symbols are real.

The quantum number quadruple $(j_1j_2j_3s)$ will be said to be admissible if, by reducing the product of IRs of HWs j_1 and j_2, the reduced states of HW $\bar{j}_3$ and of label s can be constructed. From the discussion in I, § 5, it follows that the quadruples remain admissible for even permutations of the j arguments, for odd permutations of the j arguments accompanied by the conjugation of the s argument, and for conjugation of all arguments. (The papers by Derome and Sharp (1965) and Butler (1975) will be referred to as DS and B, respectively.)

3. The s-classified SU(3) $3j$-symbols

Assuming $(j_1j_2j_3s)$ to be admissible quadruples, we define the s-classified $3j$-symbols by (cf DS (2.1) and B (5.1))

$$\begin{pmatrix} j_1 & j_2 & j_3 & s\\ m_1 & m_2 & m_3 & \end{pmatrix} = (j_1m_1j_2m_2|\bar{j}_3\bar{m}_3s)\frac{(-1)^{\bar{j}_3+\bar{m}_3}}{\sqrt{|\bar{j}_3|}}. \qquad (3.1)$$

As a result of the CGC symmetries (2.2), the s-classified $3j$-symbols satisfy the symmetry relations:

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 & s\\ m_1 & m_2 & m_3 & \end{pmatrix} &= \begin{pmatrix} j_2 & j_1 & j_3 & \bar{s}\\ m_2 & m_1 & m_3 & \end{pmatrix}(-1)^{j_1+j_2+j_3}\\ &= \begin{pmatrix} j_1 & j_3 & j_2 & \bar{s}\\ m_1 & m_3 & m_2 & \end{pmatrix}(-1)^{j_1+j_2+j_3}\\ &= \begin{pmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & \bar{s}\\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3 & \end{pmatrix}(-1)^{j_1+j_2+j_3}. \qquad (3.2)\end{aligned}$$

Thus, differently from what happens in the SU(3) theory discussed in Derome (1967) and Butler (1975), the s-classified $3j$-symbols linked by the basic symmetry relations of transposition and conjugation are those referring just to opposite multiplicity labels, the s-classified $3j$ transposition and conjugation matrices being (cf DS (2.8) and (4.1))

$$M(12,3)_s^{s'} = M(1,23)_s^{s'} = \delta_{s\bar{s}'}(-1)^{j_1+j_2+j_3} \qquad A(123)_{ss'} = \delta_{s\bar{s}'}.$$

The s-classified $3j$-symbols are invariant for even permutations of the jm columns. For odd permutations of the columns accompanied by the conjugation of the s argument, they are multiplied by $(-1)^\sigma$, where $\sigma = j_1+j_2+j_3$; the conjugation of all arguments also leads to the factor $(-1)^\sigma$. When multiplied by $\sqrt{|j_3|}$, the $3j$-symbols of fixed j_1 and j_2 form an orthonormal matrix with the matrix indices (m_1m_2) and (j_3m_3s). As indicated by (3.1), the s-classified SU(3) $1j$-symbol is (see DS (3.1) or B (5.10))

$$\begin{pmatrix} j \\ mm' \end{pmatrix} = \delta_{m\bar{m}'}(-1)^{j+m}. \tag{3.3}$$

4. The s-classified SU(3) $6j$-symbols

Assuming the quadruples $(j_1j_2j_3s_1)$, $(\bar{j}_1j_5\bar{j}_6s_2)$, $(\bar{j}_4\bar{j}_2j_6s_3)$ and $(j_4\bar{j}_5\bar{j}_3s_4)$ to be admissible, we introduce the s-classified $6j$-symbols by the definition (cf DS (5.1) and B (9.6))

$$\begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_4 & j_5 & j_6 & \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix}$$

$$= \sum_{\text{all } m} (-1)^{\sum_{i=1}^6 (j_i+m_i)} \begin{pmatrix} j_1 & j_2 & j_3 & s_1 \\ m_1 & m_2 & m_3 & \end{pmatrix} \begin{pmatrix} \bar{j}_1 & j_5 & \bar{j}_6 & s_2 \\ \bar{m}_1 & m_5 & \bar{m}_6 & \end{pmatrix}$$
$$\times \begin{pmatrix} \bar{j}_4 & \bar{j}_2 & j_6 & s_3 \\ \bar{m}_4 & \bar{m}_2 & m_6 & \end{pmatrix} \begin{pmatrix} j_4 & \bar{j}_5 & \bar{j}_3 & s_4 \\ m_4 & \bar{m}_5 & \bar{m}_3 & \end{pmatrix}. \tag{4.1}$$

As a consequence of the $3j$ symmetries, the s-classified $6j$-symbols are invariant for the rearrangements of the j arguments known from the SU(2) case. In our case, however, each rearrangement has to be accompanied by an additional transformation in order to make all four quadruples of the new symbol admissible. The transformation consists of conjugating some of the arguments and of permutating the s arguments; whereas the permutation is uniquely specified by the rearrangement, the conjugation can be performed in two ways: either all s arguments are conjugated or none. The total number of the symmetry related $6j$-symbols of the same value is thus 48. As the basic symmetry relations one can use those given by (cf DS, theorems 1 and 2, and B (9.7)-(9.9))

$$\begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_4 & j_5 & j_6 & \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix} = \begin{Bmatrix} \bar{j}_2 & \bar{j}_1 & \bar{j}_3 & \\ j_5 & j_4 & j_6 & \\ s_1 & s_3 & s_2 & s_4 \end{Bmatrix} = \begin{Bmatrix} \bar{j}_1 & \bar{j}_3 & \bar{j}_2 & \\ j_4 & j_6 & j_5 & \\ s_1 & s_2 & s_4 & s_3 \end{Bmatrix}$$
$$= \begin{Bmatrix} j_4 & \bar{j}_5 & \bar{j}_3 & \\ j_1 & \bar{j}_2 & \bar{j}_6 & \\ s_4 & s_3 & s_2 & s_1 \end{Bmatrix} = \begin{Bmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & \\ \bar{j}_4 & \bar{j}_5 & \bar{j}_6 & \\ \bar{s}_1 & \bar{s}_2 & \bar{s}_3 & \bar{s}_4 \end{Bmatrix}. \tag{4.2}$$

Proceeding as in the SU(2) case one finds that the $6j$-symbols fulfil the orthogonality relations and the sum rules expressed by (cf DS, theorems 4 and 5, and B (9.10) and (9.11))

$$\sum_{j_6 s_2 s_3} |j_6| \begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_4 & j_5 & j_6 & \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3' & \\ j_4 & j_5 & j_6 & \\ s_1' & s_2 & s_3 & s_4' \end{Bmatrix} = \delta_{j_3 j_3'} \delta_{s_1 s_1'} \delta_{s_4 s_4'} |j_3|^{-1}$$

$$\sum_{jss'} (-1)^{j+j_3+j_6} |j| \begin{Bmatrix} j_1 & j_4 & j & \\ \bar{j}_5 & \bar{j}_2 & j_3 & \\ s & \bar{s}_1 & s_4 & \bar{s}' \end{Bmatrix} \begin{Bmatrix} j_1 & j_4 & j & \\ j_2 & j_5 & j_6 & \\ s & s_2 & \bar{s}_3 & s' \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_4 & j_5 & j_6 & \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix}.$$

Whenever one of the j arguments vanishes, the $6j$-symbol reduces to a simple algebraic expression; if, for example, $j_6 = 0$ (and, necessarily, $j_5 = j_1$, $j_4 = \bar{j}_2$, $s_1 = s_4$, $s_2 = 2f_3(\bar{j}_1)$ and $s_3 = 2f_3(j_2)$), then

$$\begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_4 & j_5 & 0 & \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix} = \frac{(-1)^{j_1+j_2+j_3}}{(|j_1||j_2|)^{1/2}}. \tag{4.3}$$

Up to a factor, the s-classified SU(3) $6j$-symbols are identical with the recoupling coefficients linking different s-classified reduced states of products of three SU(3) IRS; explicitly (cf B (9.13))

$$\langle((j_1 j_2) j_{12}\ j_3) jm s_{12} s | (j_1 (j_2 j_3) j_{23}) jm s_{23} s' \rangle$$

$$= (-1)^{j_1+j_2+j_3+j} (|j_{12}||j_{23}|)^{1/2} \begin{Bmatrix} j_1 & j_2 & \bar{j}_{12} & \\ j_3 & j & j_{23} & \\ s_{12} & s' & s_{23} & s \end{Bmatrix}. \tag{4.4}$$

5. The s-classified SU(3) $9j$-symbols

Assuming all quadruples in (the first three) columns and rows to be admissible, we define the s-classified $9j$-symbols by (see DS, p 1590 and B (10.2))

$$\begin{Bmatrix} j_1 & j_2 & j_3 & s_1 \\ j_4 & j_5 & j_6 & s_2 \\ j_7 & j_8 & j_9 & s_3 \\ s_4 & s_5 & s_6 & \end{Bmatrix}$$

$$= \sum_{\text{all } m} \begin{pmatrix} j_1 & j_2 & j_3 & s_1 \\ m_1 & m_2 & m_3 & \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 & s_2 \\ m_4 & m_5 & m_6 & \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 & s_3 \\ m_7 & m_8 & m_9 & \end{pmatrix}$$

$$\times \begin{pmatrix} j_1 & j_4 & j_7 & s_4 \\ m_1 & m_4 & m_7 & \end{pmatrix} \begin{pmatrix} j_2 & j_5 & j_8 & s_5 \\ m_2 & m_5 & m_8 & \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 & s_6 \\ m_3 & m_6 & m_9 & \end{pmatrix}. \tag{5.1}$$

As a consequence of the $3j$ symmetries, the s-classified $9j$-symbols are invariant for even permutations of the first three columns or rows, for reflection in the main diagonal, and for conjugation of all arguments. For odd permutations of the first three columns (rows) accompanied by the conjugation of the last one they are multiplied by $(-1)^{\sigma}$, where σ is the sum of all j arguments. The total number of the symmetry related $9j$-symbols of the same magnitude is 144. In particular,

$$\begin{Bmatrix} j_1 & j_2 & j_3 & s_1 \\ j_4 & j_5 & j_6 & s_2 \\ j_7 & j_8 & j_9 & s_3 \\ s_4 & s_5 & s_6 & \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 & \bar{s}_1 \\ j_5 & j_4 & j_6 & \bar{s}_2 \\ j_8 & j_7 & j_9 & \bar{s}_3 \\ s_5 & s_4 & s_6 & \end{Bmatrix}(-1)^\sigma = \begin{Bmatrix} j_1 & j_3 & j_2 & \bar{s}_1 \\ j_4 & j_6 & j_5 & \bar{s}_2 \\ j_7 & j_9 & j_8 & \bar{s}_3 \\ s_4 & s_6 & s_5 & \end{Bmatrix}(-1)^\sigma$$

$$= \begin{Bmatrix} j_1 & j_4 & j_7 & s_4 \\ j_2 & j_5 & j_8 & s_5 \\ j_3 & j_6 & j_9 & s_6 \\ s_1 & s_2 & s_3 & \end{Bmatrix} = \begin{Bmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & \bar{s}_1 \\ \bar{j}_4 & \bar{j}_5 & \bar{j}_6 & \bar{s}_2 \\ \bar{j}_7 & \bar{j}_8 & \bar{j}_9 & \bar{s}_3 \\ \bar{s}_4 & \bar{s}_5 & \bar{s}_6 & \end{Bmatrix}. \tag{5.2}$$

In the same way as in the SU(2) case one can show that the s-classified $9j$-symbols are expressible in terms of the s-classified $6j$-symbols (cf B (10.3)):

$$\begin{Bmatrix} j_1 & j_2 & j_3 & s_1 \\ j_4 & j_5 & j_6 & s_2 \\ j_7 & j_8 & j_9 & s_3 \\ s_4 & s_5 & s_6 & \end{Bmatrix} = \sum_{jss's''} |j| \begin{Bmatrix} j_1 & j_2 & j_3 & \\ \bar{j}_6 & j_9 & j & \\ s_1 & \bar{s} & s' & \bar{s}_6 \end{Bmatrix} \begin{Bmatrix} j_4 & j_5 & j_6 & \\ j_2 & j & \bar{j}_8 & \\ s_2 & s'' & \bar{s}_5 & \bar{s}' \end{Bmatrix} \begin{Bmatrix} j_7 & j_8 & j_9 & \\ j & \bar{j}_1 & j_4 & \\ s_3 & \bar{s}_4 & \bar{s}'' & s \end{Bmatrix} \tag{5.3}$$

and that they satisfy the orthogonality relations

$$\sum_{\substack{j_7 j_8 \\ s_3 s_4 s_5}} |j_7||j_8| \begin{Bmatrix} j_1 & j_2 & j_3 & s_1 \\ j_4 & j_5 & j_6 & s_2 \\ j_7 & j_8 & j_9 & s_3 \\ s_4 & s_5 & s_6 & \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3' & s_1' \\ j_4 & j_5 & j_6' & s_2' \\ j_7 & j_8 & j_9 & s_3 \\ s_4 & s_5 & s_6' & \end{Bmatrix} = \delta_{j_3 j_3'}\delta_{j_6 j_6'}\delta_{s_1 s_1'}\delta_{s_2 s_2'}\delta_{s_6 s_6'}(|j_3||j_6|)^{-1}$$

and fulfil the sum rules

$$\sum_{\substack{jj' \\ ss's''}} (-1)^{j'+j_6+j_8}|j||j'| \begin{Bmatrix} j_1 & j_2 & j_3 & s_1 \\ j_5 & j_4 & j_6 & \bar{s}_2 \\ j & j' & j_9 & s'' \\ s & \bar{s}' & s_6 & \end{Bmatrix} \begin{Bmatrix} j_1 & j_5 & j & s \\ j_4 & j_2 & j' & s' \\ j_7 & j_8 & j_9 & s_3 \\ s_4 & \bar{s}_5 & s'' & \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 & s_1 \\ j_4 & j_5 & j_6 & s_2 \\ j_7 & j_8 & j_9 & s_3 \\ s_4 & s_5 & s_6 & \end{Bmatrix}.$$

For one of the j arguments vanishing, the $9j$-symbol reduces to a $6j$-symbol up to a factor; if, for example, $j_9 = 0$ (and, necessarily, $j_6 = \bar{j}_3$, $j_8 = \bar{j}_7$, $s_3 = 2f_3(j_7)$ and $s_6 = 2f_3(j_3)$), then

$$\begin{Bmatrix} j_1 & j_2 & j_3 & s_1 \\ j_4 & j_5 & j_6 & s_2 \\ j_7 & j_8 & 0 & s_3 \\ s_4 & s_5 & s_6 & \end{Bmatrix} = \frac{(-1)^{j_2+j_3+j_4+j_7}}{(|j_3||j_7|)^{1/2}} \begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_5 & \bar{j}_4 & j_7 & \\ s_1 & \bar{s}_4 & s_5 & \bar{s}_2 \end{Bmatrix}. \tag{5.4}$$

The s-classified SU(3) $9j$-symbols and the recoupling coefficients linking different s-classified reduced states of products of four SU(3) IRS are related by the formula

$$\langle ((j_1 j_2)j_{12}(j_3 j_4)j_{34})jms_{12}s_{34}s | ((j_1 j_3)j_{13}(j_2 j_4)j_{24})jms_{13}s_{24}s' \rangle$$

$$= (|j_{12}||j_{34}||j_{13}||j_{24}|)^{1/2} \begin{Bmatrix} j_1 & j_2 & \bar{j}_{12} & s_{12} \\ j_3 & j_4 & \bar{j}_{34} & s_{34} \\ \bar{j}_{13} & \bar{j}_{24} & j & \bar{s}' \\ s_{13} & s_{24} & \bar{s} & \end{Bmatrix}. \tag{5.5}$$

The question of possible higher $3nj$ symmetries (cf Regge 1958, 1959) requires a special investigation.

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